Fourier Analysis

Note Title 2/16/2020

Review: Good kernels and the Convergence Theorem

Def. A sequence $(K_n)_{n=1}^{\infty} \subset \mathcal{R}[-\pi,\pi]$ is said to be a good kernel if

(1) $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1$ for all $n \in \mathbb{N}$

(2) $\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M$ for all $n \in \mathbb{N}$, where M is a constant

(3) $\forall\ \delta > 0$,

$$\int_{\delta < |x| \leq \pi} |K_n(x)|\,dx \to 0 \quad \text{as } n \to \infty.$$

Similarly, we can define "good kernel" for a family of integrable functions

$$(K_t)_{t \in (a,b)}$$

On the circle as $t \to t_0$

Thm (Convergence Thm)

Let $(K_t(x))_{t\in(a,b)}$ be a good kernel on the circle as $t \to t_0$. Let $f \in \mathcal{R}[-\pi,\pi]$.

Then

① $f * K_t(x) \to f(x)$ as $t \to t_0$ if f is continuous at x.

② If f is continuous on the circle, then

$$f * K_t(x) \rightrightarrows f(x) \text{ as } t \to t_0$$

on the circle.

Examples of good kernels:

Fejér kernel and Poisson kernel

1. Fejér kernel

Let $D_N(x) = \sum_{|n| \leq N} e^{inx}$ denote the N-th Dirichlet kernel.

The N-th Fejer kernel:

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}$$

$$= \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) e^{inx}$$

$$= \frac{\sin^2\left(\frac{N}{2}x\right)}{N \sin^2(x/2)}.$$

As was proved, $(F_N)_{N=1}^{\infty}$ is a good kernel as $N \to \infty$.

Corollary (Fejér's Thm)

Let $f \in \mathcal{R}[-\pi, \pi]$. Then

(1) $f * F_N(x) \to f(x)$ as $N \to \infty$
if f is cts at x.

(2) If f is cts on the circle, then

$f * F_N(x) \rightrightarrows f(x)$ on the circle.

Remark: Suppose $f \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$,

then $$f * K_N(x) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N}\right) a_n e^{inx}.$$

Write also
$$\sigma_N(f)(x) = f * K_N(x)$$
and call it
"the N-th Cesaro mean of the Fourier series of f."

Corollary. Continuous functions on the circle can be unif. approximated by trigonometric polynomials.

That is, if f is cts on $[-\pi, \pi]$ and $f(\pi) = f(-\pi)$, then $\forall\ \varepsilon > 0$, $\exists$ a trigonometric polynomial P such that

$$|f(x) - P(x)| < \varepsilon \quad \text{for all } x \in [-\pi, \pi].$$

Pf. Notice that $\sigma_N(f)(x) \rightrightarrows f(x)$ and $\sigma_N(f)$ are trigonometric polynomials. ▨

Corollary. Let f be a continuous function on the circle so that

$$\hat{f}(n) = 0 \quad \text{for all } n \in \mathbb{Z}.$$

Then $f \equiv 0$.

Pf. Since $\hat{f}(n) \equiv 0$,

$$\sigma_N(f) \equiv 0.$$

However,

$$\sigma_N(f) \rightrightarrows f \quad \text{as } N \to \infty$$

So $f \equiv 0$.

2. Poisson kernel on the circle:

$$\left(P_r(x)\right)_{0\leq r<1} \quad \text{where}$$

$$P_r(x) = \frac{1-r^2}{1-2r\cos x + r^2}$$

$$= \sum_{n=-\infty}^{\infty} r^{|n|} e^{inx}.$$

As was proved, $(P_r)_{0\leq r<1}$ is a good kernel as $r \to 1$.

Corollary. Let $f \in \mathcal{R}[-\pi, \pi]$. Then

$$\lim_{r \to 1} f * P_r(x) = f(x) \text{ if } f \text{ is cts at } x.$$

Moreover

$$f * P_r(x) \rightrightarrows f(x) \quad \text{on the circle}$$

if f is cts on the circle.

Remark: If $f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$,

then

$$f * P_r(x) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{inx}.$$

We also write

$$A_r(f)(x) = f * P_r(x)$$

and call it the Abel mean of the Fourier series of f.